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Duality between integrable Stäckel systems

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Abstract. Canonical transformations of the extended phase space are applied to the integrable Stäckel systems. All these transformations may be associated with an ambiguity of the Abel map on the corresponding hyperelliptic curve. For some Stäckel systems with two degrees of freedom the 2×2 Lax representations and the dynamical r-matrix algebras are constructed.

1. Introduction

On the $2n$-dimensional symplectic manifold $\mathcal{M}$ (phase space) with coordinates $\{p_j, q_j\}_{j=1}^n$ let us consider a Hamiltonian system with some Hamilton function $H(p, q)$. To consider canonical transformations of the time t, we add a new coordinate $q_{n+1} = t$ with the corresponding momenta $p_{n+1} = H$ to the initial phase space $\mathcal{M}$. The resulting $(2n + 2)$-dimensional space $\mathcal{M}_E$ is the so-called extended phase space of the given Hamiltonian system.

By definition canonical transformations of the extended phase space $\mathcal{M}_E$ preserve the differential form $\Omega = \sum_{j=1}^{n+1} p_j \, \mathrm{d}q_j$ and the Hamilton–Jacobi equation

$$\frac{\partial \mathcal{S}}{\partial t} + H = 0. \tag{1.1}$$

Such transformations have the following form:

$$t \mapsto \tilde{t} \qquad \mathrm{d}\tilde{t} = v^{-1}(p, q)\, \mathrm{d}t \tag{1.2}$$

$$H \mapsto \tilde{H} = v(p, q)H. \tag{1.3}$$

It is known that any canonical transformation of the initial phase space $\mathcal{M}$ maps any integrable system into the other integrable system. We do not have a regular way to obtain a canonical transformation of the extended phase space $\mathcal{M}_E$, which maps a given integrable system into the other integrable system.

In this paper for the Stäckel system we introduce canonical transformations of the extended phase space associated with the ambiguity of the Abel map on the hyperelliptic curve. For some Stäckel systems we propose Lax pairs and r-matrix algebras. As examples the Henon–Heiles systems, integrable Holt potentials and the integrable deformations of the Kepler problem are discussed in detail.

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2. Duality between the Stäckel systems

The systems associated with the name of Stäckel [3] are holonomic systems on the phase space $\mathbb{R}^{2n}$ equipped with the canonical variables $\{p_j, q_j\}_{j=1}^n$, with the standard symplectic structure Ω_n and with the following Poisson brackets:

$$\Omega_n = \sum_{j=1}^n \mathrm{d}p_j \wedge \mathrm{d}q_j \qquad \{p_j, q_k\} = \delta_{jk}. \tag{2.1}$$

The non-degenerate $n \times n$ Stäckel matrix $\boldsymbol{S}$, whose j column s_{kj} depends only on q_j

$$\det \boldsymbol{S} \neq 0 \qquad \frac{\partial s_{kj}}{\partial q_m} = 0 \qquad j \neq m$$

defines n functionally independent integrals of motion

$$I_k = \sum_{j=1}^n c_{jk}\big(p_j^2 + U_j\big) \qquad c_{jk} = \frac{\boldsymbol{S}_{kj}}{\det \boldsymbol{S}} \tag{2.2}$$

which are quadratic in momenta. Here $\boldsymbol{C} = [c_{ik}]$ denotes an inverse matrix to $\boldsymbol{S}$ and $\boldsymbol{S}_{kj}$ is the cofactor of the element s_{kj}. The common level surface of the integrals (2.2)

$$M_\alpha = \big\{z \in \mathbb{R}^{2n}\colon I_k(z) = \alpha_k, k = 1, \dots, n\big\} \tag{2.3}$$

is diffeomorphic to the n-dimensional real torus and one immediately obtains

$$p_j^2 = \left(\frac{\partial \mathcal{S}}{\partial q_j}\right)^2 = \sum_{k=1}^n \alpha_k s_{kj}(q_j) - U_j(q_j). \tag{2.4}$$

Here $\mathcal{S}(q_1 \dots, q_n)$ is a reduced action function [4].

The corresponding Hamilton–Jacobi equation on M_α (1.1) admits the variable separation

$$\mathcal{S}(q_1 \dots, q_n) = \sum_{j=1}^n \mathcal{S}_j(q_j) \qquad \mathcal{S}_j(q_j) = \int \sqrt{F_j(q_j)}\, \mathrm{d}q_j. \tag{2.5}$$

Here the functions $F_j(\lambda)$ depend on n parameters $\{\alpha_k\}_{k=1}^n$

$$F_j(\lambda) = \sum_{k=1}^n \alpha_k s_{kj}(\lambda) - U_j(\lambda).$$

By definition the first integral $I_1 = H$ is the Hamilton function associated with the time t. Hence, coordinates $q_j(t, \alpha_1, \dots, \alpha_n)$ are determined from the equation explicitly depending on time

$$\sum_{j=1}^n \int_{\gamma_0(p_0,q_0)}^{\gamma_j(p_j,q_j)} \frac{s_{1j}(\lambda)\, \mathrm{d}\lambda}{\sqrt{\sum_{k=1}^n \alpha_k s_{1j}(\lambda) - U_j(\lambda)}} = \beta_1 = t \tag{2.6}$$

and from $n-1$ other equations

$$\sum_{j=1}^n \int_{\gamma_0(p_0,q_0)}^{\gamma_j(p_j,q_j)} \frac{s_{kj}(\lambda)\, \mathrm{d}\lambda}{\sqrt{\sum_{k=1}^n \alpha_k s_{kj}(\lambda) - U_j(\lambda)}} = \beta_k \qquad k = 2, \dots, n. \tag{2.7}$$

The solution of the problem is thus reduced to solving a sequence of one-dimensional problems.

Now we turn to the canonical change of the time and prove the following.

Proposition 1. *If the two Stäckel matrices $\boldsymbol{S}$ and $\tilde{\boldsymbol{S}}$ can be distinguished in the first row only*

$$s_{kj} = \tilde{s}_{kj} \qquad k \neq 1$$

the corresponding Stäckel systems with the following Hamilton functions:

$$\tilde{H} = v(q)H \qquad v(q) = \frac{\det \boldsymbol{S}(q_1, \ldots, q_n)}{\det \tilde{\boldsymbol{S}}(q_1, \ldots, q_n)} \tag{2.8}$$

are related by a canonical change of the time.

In fact, the Hamilton functions H and $\tilde{H}$ obey equation (2.8), which follows from the definition of the Hamiltonian,

$$H = \sum_{j=1}^{n} c_{j1}\big(p_j^2 + U_j(q_j)\big) \tag{2.9}$$

and the definition of the inverse matrix with entries

$$c_{j1} = \frac{\boldsymbol{S}_{1j}}{\det \boldsymbol{S}} = \frac{1}{\det \boldsymbol{S}} \frac{\partial \det \boldsymbol{S}}{\partial \boldsymbol{s}_{1j}}.$$

In contrast with the general coupling constant metamorphosis [1] equation (2.8) is independent on any constant in the potential U.

As an example, let us consider three matrices

$$\boldsymbol{S} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \qquad \tilde{\boldsymbol{S}} = \begin{pmatrix} q_1 & q_2 \\ 1 & -1 \end{pmatrix} \qquad \widehat{\boldsymbol{S}} = \begin{pmatrix} q_1^2 & q_2^2 \\ 1 & -1 \end{pmatrix}. \tag{2.10}$$

The corresponding Hamiltonians H, $\tilde{H}$ and $\widehat{H}$ defined by (2.9) are dual

$$\begin{aligned} \tilde{H} &= \tfrac{1}{2}(q_1 + q_2)^{-1} H \\ \widehat{H} &= \tfrac{1}{2}(q_1^2 + q_2^2)^{-1} H = \frac{q_1 + q_2}{q_1^2 + q_2^2} \tilde{H}. \end{aligned} \tag{2.11}$$

For any function $\xi(q)$ depending on coordinates only one obtains the following transformations of the time:

$$\frac{\mathrm{d}\xi(q)}{\mathrm{d}\tilde{t}} = \{\tilde{H}, \xi(q)\} = \tfrac{1}{2}(q_1 + q_2)^{-1}\{H, \xi(q)\} = \frac{1}{2(q_1 + q_2)} \frac{\mathrm{d}\xi(q)}{\mathrm{d}t}. \tag{2.12}$$

The uniform cubic potential

$$U(q_j) = 2\alpha^2 q_j^3 + \beta q_j^2 + \gamma q_j + \delta \tag{2.13}$$

gives rise to the Hamiltonian H

$$H = \tfrac{1}{4}(p_1^2 + p_2^2) + \alpha^2(q_1^3 + q_2^3) + \tfrac{1}{2}\beta(q_1^2 + q_2^2) + \tfrac{1}{2}\gamma(q_1 + q_2) + \delta. \tag{2.14}$$

By using the canonical transformation

$$\begin{aligned} q_1 &= \tfrac{1}{2}(x + y) \qquad & p_1 &= p_x + p_y \\ q_2 &= \tfrac{1}{2}(x - y) \qquad & p_2 &= p_x - p_y \end{aligned} \tag{2.15}$$

for the first system, the more complicated transformation

$$\begin{aligned} q_1 &= \frac{3}{4}x^{2/3} + \frac{p_y}{3\alpha} \qquad & p_1 &= p_x x^{1/3} - \frac{3\alpha}{2} y \\ q_2 &= \frac{3}{4}x^{2/3} - \frac{p_y}{3\alpha} \qquad & p_2 &= p_x x^{1/3} + \frac{3\alpha}{2} y \end{aligned}$$

for the system associated with $\tilde{S}$ and the following change of variables in the third case:

$$q_1 = \sqrt{x} - \sqrt{y} \qquad p_1 = p_x\sqrt{x} - p_y\sqrt{y}$$
$$q_2 = -\mathrm{i}(\sqrt{x} + \sqrt{y}) \qquad p_2 = \mathrm{i}(p_x\sqrt{x} + p_y\sqrt{x})$$

one obtains the Hamilton functions in the natural form

$$H = \tfrac{1}{2}(p_x^2 + p_y^2) + \tfrac{1}{4}\alpha^2 x(x^2 + 3y^2) + \tfrac{1}{4}\beta(x^2 + y^2) + \tfrac{1}{2}\gamma x + \delta$$
$$\tilde{H} = \tfrac{1}{2}(p_x^2 + p_y^2) + \tfrac{9}{8}\alpha^2 x^{-2/3}(\tfrac{3}{4}x^2 + y^2) + \delta x^{-2/3} + \tfrac{3}{4}\gamma \qquad \text{only by} \quad \beta = 0$$
$$\widehat{H} = \frac{1}{2}p_x p_y - \frac{\beta}{2}\frac{1}{\sqrt{xy}} + \frac{\gamma}{4}\left(\frac{1+\mathrm{i}}{\sqrt{x}} - \frac{1-\mathrm{i}}{\sqrt{y}}\right) + \delta \qquad \text{by} \quad \alpha = 0$$

at $\beta = 0$ for the second case.

The system with the first Hamiltonian H is the so-called first integrable case of the Henon–Heiles system [1]. The second Hamiltonian $\tilde{H}$ is related to the so-called Holt potential [1]. The system with the third Hamiltonian $\widehat{H}$ may be considered as an integrable deformation of the Kepler problem.

We can see that in practical circumstances the Stäckel approach is not very useful because it is usually unknown which canonical transformations have to be used in order to transform a Hamiltonian (2.9) to the natural form $H = T + V$. This problem was partially solved for the uniform systems $U_j = U$, $j = 1, \ldots, n$ with polynomial potentials by using the corresponding Lax pairs [5]. In the following sections we shall restrict our attention to the uniform Stäckel systems associated with the one hyperelliptic curve $\mathcal{C} = \mathcal{C}_j$ (2.4).

3. Duality and the Abel map

Let us briefly recall some necessary facts about the Abel map and the inverse Jacobi problem. The set of point $\mathcal{C}(z, \lambda)$ satisfying

$$\mathcal{C}\colon \quad z^2 = F(\lambda) = \sum_{k=0}^{2g+1} e_k \lambda^k = \prod_{j=1}^{2g+1}(\lambda - \lambda_j) \tag{3.1}$$

is a model of a plane hyperelliptic curve of genus g. Here $F(\lambda)$ is polynomial without multiple zeros. Let us denote by $\mathrm{Div}(\mathcal{C})$ the Abelian divisor group and denote by $J(\mathcal{C})$ the Jacobian of the curve $\mathcal{C}$. The Abel map puts into correspondence the point $D \in \mathrm{Div}(\mathcal{C})$ and the point $\boldsymbol{u} \in J(\mathcal{C})$ [6, 7]

$$\mathcal{U}\colon \quad \mathrm{Div}(\mathcal{C}) \to J(\mathcal{C}). \tag{3.2}$$

The set of all effective divisors $D = \gamma_1 + \cdots + \gamma_n$ (γ_j may be not mutually distinct) of deg n of $\mathcal{C}$ is called the nth symmetric product of $\mathcal{C}$, and is denoted by $\mathcal{C}^{(n)} = S^n\mathcal{C}$. The $C^{(n)}$ can be identified with the set of all unordered n-tuples $\{\gamma_1, \ldots, \gamma_n\}$, where γ_j are arbitrary elements of $\mathcal{C}$. Now consider restriction of the Abel map (3.2) to $\mathcal{C}^{(n)}$

$$\mathcal{U}\colon \quad \mathcal{C}^{(n)} \to J(\mathcal{C}) \tag{3.3}$$

where

$$\mathcal{U}(\gamma_1, \gamma_2 \ldots, \gamma_n) = \mathcal{U}(\gamma_1) + \mathcal{U}(\gamma_2) + \cdots + \mathcal{U}(\gamma_n).$$

According to the Abel–Jacobi theorem this map is surjective and generically injective if $n = g$ only [6, 7]. If $n \neq g$ the Abel map is either not uniquely defined or degenerate.

The corresponding Stäckel system either has a dual system associated with the same curve or is a superintegrable system [5].

Suppose that the point $D = \gamma_1 + \cdots + \gamma_n$, $n \leqslant g$ belongs to $\mathcal{C}^{(n)}$. The differential of the Abel–Jacobi map (3.3) at the point D is a linear mapping from the tangent space $T_D(\mathcal{C}^{(n)})$ of $\mathcal{C}^{(n)}$ at the point D into the tangent space $T_{\mathcal{U}(D)}(J(\mathcal{C}))$ of $J(\mathcal{C})$ at the point $\mathcal{U}(D)$

$$\mathcal{U}_D^*\colon \quad T_D(C^{(n)}) \to T_{\mathcal{U}(D)}(J(\mathcal{C})).$$

Now suppose that D is a generic divisor and x_j is a local coordinate on $\mathcal{C}$ near the point γ_j. Then $(x_1, \ldots, x_n)$ yields a local coordinate system near the point D in $\mathcal{C}^{(n)}$. Let dw_k $(k = 1, \ldots, g)$ be a basis for a space $\mathcal{H}_1(\mathcal{C})$ of holomorphic differentials on $\mathcal{C}$, and near γ_j

$$\mathrm{dw}_k = \phi_{kj}(x_j)\,\mathrm{d}x_j \tag{3.4}$$

where $\phi_{kj}(x_j)$ is holomorphic. The Abel–Jacobi map $\mathcal{U}$ can be expressed near D as

$$\mathcal{U}(z_1, \ldots, z_n) = \left(\sum_{j=1}^{n} \int_{\gamma_0}^{x_j} \phi_{1j}(x_j)\,\mathrm{d}x_j, \ldots, \sum_{j=1}^{n} \int_{\gamma_0}^{x_j} \phi_{gj}(x_j)\,\mathrm{d}x_j \right).$$

Hence

$$\mathcal{U}_D^* = \begin{pmatrix} \phi_{11}(\gamma_1) & \cdots & \phi_{g1}(\gamma_1) \\ \vdots & \ddots & \vdots \\ \phi_{1n}(\gamma_n) & \cdots & \phi_{gn}(\gamma_n) \end{pmatrix} \tag{3.5}$$

is the so-called Brill–Noether matrix [8]. Henceforth, we shall restrict our attention to the special divisors D_s, such that coefficients in the expansion (3.4) are independent on the point γ_j

$$\mathrm{dw}_k = \phi_k(x_j)\,\mathrm{d}x_j.$$

In this case all the rows of the $n \times g$ homogeneous Brill–Noether matrix depend on local coordinates $\{x_1, \ldots, x_n\}$ identically.

The Jacobi inversion problem (2.7) is formulated as follows: for a given point $\boldsymbol{u} = (\beta_1, \beta_2 \ldots, \beta_n) \in J(\mathcal{C})$ find n points $\gamma_1, \gamma_2 \ldots, \gamma_n$ on the genus g Riemann surface $\mathcal{C}$ such that

$$\sum_{k=1}^{g} \int_{\gamma_0}^{\gamma_k} \mathrm{dw}_j = \beta_j \qquad j = 1, \ldots, n. \tag{3.6}$$

Here we shall tacitly assume that the base point $\gamma_0 \in \mathcal{C}$ has already been fixed [6].

If $n = g$ for almost all points $\boldsymbol{u} \in J(\mathcal{C})$ the solution $D = \gamma_1 + \cdots + \gamma_n$ exist and is uniquely determined by system (3.6) (for the unordered set of points γ_j) [6]. However, if the degree $n < g$ of the symmetric product $\mathcal{C}^{(n)}$ is less than genus g of $\mathcal{C}$, the Abel map shows a lack of uniqueness. In this case we can propose that two different points $\boldsymbol{u}, \tilde{\boldsymbol{u}} \in J(\mathcal{C})$ have one Abel preimage $\{\gamma_1, \ldots, \gamma_n\} \in \mathcal{C}^{(n)}$.

The Abel preimage of the point $\boldsymbol{u} \in J(\mathcal{C})$ is given by the set $\{(p_1, q_1), \ldots, (p_n, q_n)\} \in \mathcal{C}^{(n)}$, where $\{q_1, \ldots, q_n\}$ are zeros of the Bolza equation [7, 9]

$$e(\lambda, \boldsymbol{u}) = \lambda^n - \lambda^{n-1}\wp_{n,n}(\boldsymbol{u}) - \lambda^{n-2}\wp_{n,n-1}(\boldsymbol{u}) - \cdots - \wp_{n,1}(\boldsymbol{u}) = 0 \tag{3.7}$$

and $\{p_1, \ldots, p_n\}$ are equal to

$$p_k = -\left.\frac{\partial e(\lambda, \boldsymbol{u})}{\partial \beta_n}\right|_{\lambda=q_k}. \tag{3.8}$$

Here the vector $\boldsymbol{u}$ belongs to the Jacobian $J(\mathcal{C})$ and $\wp_{k,j}(\boldsymbol{u})$ is the Kleinian $\wp$-function [7, 9].

Now we turn to the uniform Stäckel systems. We can regard each expression (2.4) as being defined on the genus g Riemann surface

$$\mathcal{C}: \quad y_j^2 = F(\lambda) \qquad F(\lambda) = \sum_{k=1}^{n} \alpha_k s_{kj}(\lambda) - U(\lambda) \tag{3.9}$$

which depends on the values α_k of integrals of motion. For the Stäckel systems on $\mathbb{R}^{2n}$ the minimum admissible genus g of the curve $\mathcal{C}$ is equal to $g = [(n-1)/2]$.

The nth symmetric product of $\mathcal{C}$ defines the n-dimensional Lagrangian submanifold in the complete symplectic manifold $\mathbb{R}^{2n}$

$$\mathcal{C}^{(n)}: \quad \mathcal{C}(p_1, q_1) \times \mathcal{C}(p_2, q_2) \times \cdots \times \mathcal{C}(p_n, q_n). \tag{3.10}$$

Then, the integration problem (2.6) and (2.7) for the equation of motion is reduced to the inverse Jacobi problem (3.3) on the Lagrangian submanifold (3.10). The corresponding holomorphic differentials $\delta \mathrm{w}_k$ are equal to

$$\mathrm{dw}_k = \frac{s_{kj}(\lambda)\,\mathrm{d}\lambda}{z(\lambda)}. \tag{3.11}$$

The set of these differentials either forms a basis in the space of holomorphic differentials $\mathcal{H}_1(\mathcal{C})$ [6] or may be a complement to a basis. The corresponding $n \times n$ Stäckel matrix be the $n \times n$ block of the transpose Brill–Noether matrix $\mathcal{U}_D^{*t}$.

The different blocks are determined by the dual Stäckel systems. In this case vectors differing in the first entry only

$$\boldsymbol{u} = \{t, \beta_2, \ldots, \beta_n\} \in J(\mathcal{C}) \qquad \tilde{\boldsymbol{u}} = \{\tilde{t}, \beta_2, \ldots, \beta_n\} \in J(\mathcal{C})$$

have a common Abel preimage $\{(p_1, q_1), \ldots, (p_n, q_n)\} \in \mathcal{C}^{(n)}$.

Let us consider the standard basis of holomorphic differentials in $\mathcal{H}_1(\mathcal{C})$

$$\mathrm{dw}_k = \frac{\lambda^{k-1}}{z(\lambda)}\,\mathrm{d}\lambda \qquad k = 1, \ldots, g. \tag{3.12}$$

Recall, that the derivative $\mathcal{U}_D^*$ bears a great resemblance to the canonical map $\mathcal{C} \to \mathbb{P}^{g-1}$ and, therefore, to the Veronese map $\mathbb{P}^1 \to \mathbb{P}^{g-1}$ given by a basis for the polynomial ring of degree $g-1$. With respect to the basis of $\mathcal{H}_1(\mathcal{C})$ (3.12), the Veronese map of $\mathcal{C}$ has an extremely simple expression

$$(y, \lambda) \to \lambda \to [\lambda^{g-1}, \lambda^{g-2}, \ldots, \lambda, 1].$$

By using the corresponding homogeneous Brill–Noether matrix $\mathcal{U}_D^*$ (3.5), we shall determine the Stäckel matrices as $(n \times n)$ blocks of the following $(g \times n)$ matrix:

$$\begin{pmatrix} q_1^{g-1} & q_2^{g-1} & \cdots & q_n^{g-1} \\ q_1^{g-2} & q_2^{g-2} & \cdots & q_n^{g-2} \\ \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & \cdots & 1 \end{pmatrix}. \tag{3.13}$$

Evidently, all the Stäckel matrices cannot be obtained from the homogeneous Brill–Noether matrices. For instance, the Stäckel matrices (2.10) are non-homogeneous matrices.

4. Lax representation

Henceforth, we shall restrict our attention to the basis (3.12) and the homogeneous matrix (3.13). For the corresponding Stäckel systems let us look for the Lax representation as

$$L = \begin{pmatrix} h(\lambda, p, q) & e(\lambda, q) \\ f(\lambda, p, q) & -h(\lambda, p, q) \end{pmatrix}. \tag{4.1}$$

Let us fix the hyperelliptic genus g curve $\mathcal{C}$ and the dimension of the phase space $n \leqslant g$. Then we extract the $(n \times n)$ Stäckel matrix $\boldsymbol{S}$ from the matrix (3.13) and define the Hamilton function H (2.9) with $U = 0$. Next we define the function $e(\lambda, \boldsymbol{u})$ (3.7)

$$e(\lambda, q) = \prod_{j=1}^{n} (\lambda - q_j) \tag{4.2}$$

with n zeros, which are solutions of the inverse Jacobi problem. Then we introduce the second entry of the Lax matrix as

$$h(\lambda) = -\frac{1}{2v(\lambda, q)} \frac{\mathrm{d}e(\lambda)}{\mathrm{d}t} + w(\lambda, p, q)\, e(\lambda).$$

Here the function $v(\lambda, q)$ is calculated by using the second Bolza equation (3.8)

$$h(\lambda)|_{\lambda=q_k} = p_k = \left(\frac{1}{2v} \frac{\mathrm{d}e(\lambda)}{\mathrm{d}t} \right)_{\lambda=q_k} = -\left.\frac{\partial e(\lambda)}{\partial u_n}\right|_{\lambda=q_k}. \tag{4.3}$$

Let the third entry of the Lax matrix takes the form

$$f(\lambda) = \frac{1}{v} \frac{\mathrm{d}h(\lambda)}{\mathrm{d}t}.$$

Here the single unknown function $w(\lambda, p, q)$ is determined such that the spectral curve of the Lax matrix (4.1)

$$\mathcal{C}: \quad z^2 = F(\lambda) = -\det L_0(\lambda) = h^2(\lambda) + e(\lambda) f(\lambda) \tag{4.4}$$

be the same as the initial algebraic curve $\mathcal{C}$ (2.4) by $U = 0$.

The above constructed matrix $L_0(\lambda)$ (4.1) reads as

$$L_0(\lambda) = \begin{pmatrix} -\frac{1}{2v} e_t(\lambda) + w(\lambda, p, q)\, e(\lambda) & e(\lambda) \\ \frac{1}{v} h_t(\lambda) & \frac{1}{2v} e_t(\lambda) - w(\lambda, p, q)\, e(\lambda) \end{pmatrix} \tag{4.5}$$

where

$$e_t = \frac{\mathrm{d}e(\lambda)}{\mathrm{d}t} = \{H, e(\lambda)\} \qquad h_t = \frac{\mathrm{d}h(\lambda)}{\mathrm{d}t} = \{H, h(\lambda)\}$$

obeys the Lax equation

$$\frac{\mathrm{d}L_0}{\mathrm{d}t} = \{H, L_0\} = [A_0, L_0]$$

with the second matrix

$$A_0 = v(\lambda, q) \begin{pmatrix} w(\lambda, p, q) & 1 \\ 0 & -w(\lambda, p, q) \end{pmatrix}.$$

By definition of the Lax matrix all the pairs of separation variables $\gamma_j = (p_j, q_j)$ (4.2) and (4.3) lie on the spectral curve $\mathcal{C}$ (4.4) of the matrix L_0 (4.5)

$$z^2(\gamma_j) = p_j^2 = h^2(\lambda)\big|_{\lambda=q_j} = F(\lambda = q_j) = F(\lambda)\big|_{\gamma_j}.$$

For the systems with the polynomial potential $U \neq 0$ we propose to change the entry $f(\lambda)$ in (4.5)

$$f(\lambda) = \frac{1}{v}\frac{\mathrm{d}h(\lambda)}{\mathrm{d}t} + u(\lambda, q)\, e(\lambda)$$

where we add a new function $u(\lambda, q)$ depending on coordinates only. Of course, to construct the Lax matrix here

$$L(\lambda) = \begin{pmatrix} -\dfrac{1}{2v}e_t(\lambda) + w(\lambda, p, q)\, e(\lambda) & e(\lambda) \\ \dfrac{1}{v}h_t(\lambda) + u(\lambda, q)\, e(\lambda) & \dfrac{1}{2v}e_t(\lambda) + w(\lambda, p, q)\, e(\lambda) \end{pmatrix} \tag{4.6}$$

we have to use the complete Hamiltonian with $U \neq 0$. The associated second Lax matrix reads as

$$A = A_0 + \begin{pmatrix} 0 & 0 \\ v(\lambda, q)u(\lambda, q) & 0 \end{pmatrix} = v(\lambda, q) \begin{pmatrix} w(\lambda, p, q) & 1 \\ u(\lambda, q) & -w(\lambda, p, q) \end{pmatrix}. \tag{4.7}$$

To consider the corresponding Lax equation

$$\frac{\mathrm{d}L(\lambda)}{\mathrm{d}t} = \big[A(\lambda), L(\lambda)\big]$$

we can assume that the common factor $v(\lambda, q)$ in front of the matrix A may be associated with the change of the time for the Stäckel systems.

In general, the proof of existence functions v, w and u requires an application of the method of algebraic geometry [7]. By definition of the Lax matrices $L(\lambda)$ (4.6) and $A(\lambda)$ (4.7) this problem may be reduced to the solution of the single equation

$$\frac{\mathrm{d}f(\lambda)}{\mathrm{d}t} - 2v(uh - wf) = 0 \qquad \Longleftrightarrow \qquad \frac{\mathrm{d}F(\lambda, e, v, u)}{\mathrm{d}t} = 0 \tag{4.8}$$

for the given function $e(\lambda)$ (4.2) and the given Hamiltonian H (2.9).

If we consider the lower $(n \times n)$ block of the matrix (3.13), the differentials (3.11) span a whole space $\mathcal{H}_1(\mathcal{C})$ and the Abel map is the one-to-one correspondence. In this case from equations (3.8) and (4.3) follows that

$$v_t(\lambda, q) = 0 \qquad w(\lambda, p, q) = 0.$$

If we put $v = 1$, rename $t = x$ and introduce a 'new' time variable τ, equation (4.8) is rewritten as

$$\frac{\partial u(x, \tau, \lambda)}{\partial \tau} = \left[\tfrac{1}{4}\partial_x^3 + u(\lambda)\partial_x + \tfrac{1}{2}u_x(\lambda)\right] \cdot e(\lambda) = 0 \qquad x = t. \tag{4.9}$$

This equation may be identified with the equation for the finite-band stationary solutions $\partial u(x, \tau, \lambda)/\partial \tau = 0$ of the nonlinear soliton equations. In this theory equation (4.9) is called the generating equation. For different choices of the form of $e(\lambda)$ and $u(\lambda)$, this procedure leads to different hierarchies of integrable equations, as an example for the KdV, nonlinear Schrödinger and sine–Gordon hierarchies or to the Dym hierarchy (see references in [5]).

Function $u(\lambda, q)$ in (4.9) is constructed by using the function $e(\lambda)$ (3.7)–(4.2),

$$u(\lambda, q_1, \ldots, q_n) = \left[\phi(\lambda)\, e^{-2}(\lambda)\right]_{MN}. \tag{4.10}$$

Here $\phi(\lambda)$ is a parametric function on the spectral parameter and $[\xi]_N$ are the linear combinations of the following Taylor projections:

$$[\xi]_N = \left[\sum_{k=-\infty}^{+\infty} z_k \lambda^k\right]_N \equiv \sum_{k=0}^{N} \xi_k \lambda^k \tag{4.11}$$

or the Laurent projections [5, 11].

If the differentials (3.11) span the whole space $\mathcal{H}_1(\mathcal{C})$ the corresponding Stäckel systems describe all the possible systems, which are separable in the orthogonal curvilinear coordinate systems in $\mathbb{R}^n$ [5]. Let us consider the Stäckel systems which are dual to these systems. To apply equation (2.12) to the function $e(\lambda)$ (4.2) and by using definition (4.3) one obtains

$$\begin{aligned} p_k = \tilde{h}(\lambda)\big|_{\lambda=q_k} &= \left(-\frac{1}{2\tilde{v}}\{\tilde{H}, e(\lambda)\}\right)_{\lambda=q_k} = \frac{\det \boldsymbol{S}}{\det \tilde{\boldsymbol{S}}}\left(-\frac{1}{2\tilde{v}}\{H, e(\lambda)\}\right)_{\lambda=q_k} \\ &= \frac{\det \boldsymbol{S}}{\det \tilde{\boldsymbol{S}}}\left(\frac{v}{\tilde{v}} h(\lambda)\right)_{\lambda=q_k} = p_k \frac{\det \boldsymbol{S}}{\det \tilde{\boldsymbol{S}}}\left(\frac{v}{\tilde{v}}\right)_{\lambda=q_k}. \end{aligned} \tag{4.12}$$

Recall that $v = 1$ for the integrable system with the Hamiltonian H associated with the lower $(n \times n)$ block of the matrix (3.13).

Thus, according to (4.12), below we shall consider the Stäckel systems with the following functions $v(q)$ only:

$$v(q) = \frac{\det \boldsymbol{S}(q_1, \ldots, q_n)}{\det \tilde{\boldsymbol{S}}(q_1, \ldots, q_n)}.$$

Let us briefly discuss the canonical transformation which transforms a Hamiltonian (2.9) into the natural form $H = T + V$. For integrable systems separable in the orthogonal curvilinear coordinate systems on $\mathbb{R}^n$ the Abel map is in one-to-one correspondence and $v_t = \{H, v\} = 0$. In this case we can put $v = 1$ and introduce the function $\mathcal{B}(\lambda)$,

$$\mathcal{B}^2(\lambda) = e(\lambda) \tag{4.13}$$

which was proposed in the theory of the soliton equations [10]. It allows us to rewrite the generating function of the integrals of motion,

$$F(\lambda) = -\mathcal{B}^3 \mathcal{B}_{tt} + u(\lambda, q)\, \mathcal{B}^4 \tag{4.14}$$

as a Newton equation for the function $\mathcal{B}$

$$\ddot{\mathcal{B}}(\lambda, q) = -F(\lambda, \alpha_1, \ldots, \alpha_n)\, \mathcal{B}^{-3}(\lambda, q) + u(\lambda, q)\, \mathcal{B}(\lambda, q). \tag{4.15}$$

To expand the function $\mathcal{B}(\lambda)$ on the Laurent set

$$\mathcal{B} = \sum_{j=0}^{N} x_j \lambda^j$$

it is easy to prove that the coefficients x_j obey the Newton equation of motion (4.15) (see references within [5, 10]). Here we reinterpret the coefficients of the function $F(\lambda)$ in (4.15) not as functions on the phase space, but rather as integration constants α_j (2.3).

In general, by $v_t \neq 0$ the generating function $F(\lambda) = -\det L(\lambda)$ (4.6) is equal to

$$F(\lambda) = \frac{1}{4v^2}\left(e_t^2 - 2ee_{tt}\right) + \left(\frac{v_t}{2v^2} - w\right)\frac{e_t e}{v} + \left(w^2 + \frac{u}{v}\right)e^2.$$

In this case the suitable canonical transformations, which transform any Hamiltonian (2.9) into the natural form, are unknown.

Although we cannot prove the validity of the presented Lax representation in general, this construction works for the many well known mechanical systems. In the next section we consider some two-dimensional Stäckel systems with homogeneous Stäckel matrices in detail.

5. Examples

Let us consider four orthogonal systems of coordinates on a plane: elliptic, parabolic, polar and Cartesian. The Lax matrix $L_0(\lambda)$ (4.5) by $U = 0$ is transformed to the Lax matrix $L(\lambda)$ (4.6) by $U \neq 0$ by using the outer automorphism of the space of infinite-dimensional representations of the underlying algebra $sl(2)$ [5, 11]. We shall consider the Lax representations for the geodesic motion by $U = 0$ more extensively.

5.1. Parabolic and Cartesian coordinate systems ($w(\lambda, p, q) = 0$)

Let us consider two hyperelliptic curves,

$$\begin{aligned} \mathcal{C}^{(1)}: &\quad z^2 = \prod_{i=1}^{2g+1}(\lambda - \lambda_i) \\ \mathcal{C}^{(2)}2: &\quad z^2 = \lambda^{-1}\prod_{i=1}^{2g+1}(\lambda - \lambda_i). \end{aligned} \tag{5.1}$$

If we choose the standard basis in the space of holomorphic differentials one obtains the following homogeneous matrices (3.13) for two-dimensional systems:

$$\mathcal{U}_1^{*t}(q_1, q_2) = \begin{pmatrix} q_1^{g-1} & q_2^{g-1} \\ \vdots & \vdots \\ q_1^2 & q_2^2 \\ q_1 & q_2 \\ -1 & -1 \end{pmatrix} \qquad \mathcal{U}_2^{*t}(q_1, q_2) = \begin{pmatrix} q_1^{g-2} & q_2^{g-2} \\ \vdots & \vdots \\ q_1 & q_2 \\ 1 & 1 \\ -\dfrac{1}{q_1} & -\dfrac{1}{q_2} \end{pmatrix}. \tag{5.2}$$

Different (2×2) blocks of the matrices $\mathcal{U}_j^{*t}$ determine different Stäckel systems.

Let us consider two blocks for the each matrix. So, for the curve $\mathcal{C}^{(1)}$ we shall consider the following matrices:

$$\boldsymbol{S}_1 = \begin{pmatrix} q_1 & q_2 \\ -1 & -1 \end{pmatrix} \qquad \tilde{\boldsymbol{S}}_1 = \begin{pmatrix} q_1^2 & q_2^2 \\ -1 & -1 \end{pmatrix}. \tag{5.3}$$

For the second curve $\mathcal{C}^{(2)}$ the associated Stäckel matrices are equal to

$$\boldsymbol{S}_2 = \begin{pmatrix} 1 & 1 \\ -\dfrac{1}{q_1} & -\dfrac{1}{q_2} \end{pmatrix} \qquad \tilde{\boldsymbol{S}}_2 = \begin{pmatrix} q_1 & q_2 \\ -\dfrac{1}{q_1} & -\dfrac{1}{q_2} \end{pmatrix}. \tag{5.4}$$

Introduce the Hamilton functions (2.9) by $U = 0$

$$H_0^{(1)} = \frac{p_1^2 - p_2^2}{q_1 - q_2} \qquad \tilde{H}_0^{(1)} = (q_1 + q_2)^{-1} H_0^{(1)}$$
$$H_0^{(2)} = \frac{q_1 p_1^2 - q_2 p_2^2}{q_1 - q_2} \qquad \tilde{H}_0^{(2)} = (q_1 + q_2)^{-1} H_0^{(2)}. \tag{5.5}$$

The corresponding second integrals of motion of the dual systems are related

$$\tilde{J}_0^{(k)} = J_0^{(k)} - \frac{q_1 q_2}{q_1 + q_2} H_0^{(k)} \qquad k = 1, 2.$$

The functions $e(\lambda, \boldsymbol{u})$ (3.7)

$$e_1(\lambda) = (\lambda - q_1)(\lambda - q_2) \qquad e_2(\lambda) = \frac{(\lambda - q_1)(\lambda - q_2)}{\lambda} \tag{5.6}$$

have two zeros, which are solutions of the inverse Jacobi problem (2.7) on $\mathcal{C}^{(1)}$ and $\mathcal{C}^{(2)}$, respectively.

Let us introduce new physical variables. For the first curve $\mathcal{C}^{(1)}$ equation (4.15)

$$e_1(\lambda) = (\lambda - q_1)(\lambda - q_2) = \mathcal{B}^2(\lambda) \qquad \mathcal{B}(\lambda) = \lambda - \frac{x}{2} - \frac{y}{4\lambda}$$

yields the following canonical transformation:

$$q_1 = \tfrac{1}{2}(x - \sqrt{2y}) \qquad p_1 = p_x - \sqrt{2y}\, p_y$$
$$q_2 = \tfrac{1}{2}(x + \sqrt{2y}) \qquad p_2 = p_x + \sqrt{2y}\, p_y.$$

For the second curve $\mathcal{C}^{(2)}$ the corresponding equation

$$e_2(\lambda) = \lambda^{-1}(\lambda - q_1)(\lambda - q_2) = \lambda - x - \frac{y^2}{4\lambda}$$

defines the standard parabolic coordinate system

$$q_1 = \frac{x - \sqrt{x^2 + y^2}}{2} \qquad p_1 = p_x - \frac{\sqrt{x^2 + y^2} + x}{y} p_y$$
$$q_2 = \frac{x + \sqrt{x^2 + y^2}}{2} \qquad p_2 = p_x + \frac{\sqrt{x^2 + y^2} - x}{y} p_y.$$

By $U = 0$ the Hamilton functions are given by

$$H_0^{(1)} = 4 p_x p_y \qquad H_0^{(2)} = p_x^2 + p_y^2.$$

According to (4.3) and (4.12) functions $v(q_1, q_2)$ entering in the Lax representation are equal to

$$v(\lambda, q_1, q_2) = \begin{cases} 1 & \text{for matrices} \quad \boldsymbol{S}_{1,2} \\ (q_1 + q_2)^{-1} = \dfrac{1}{x} & \text{for matrices} \quad \tilde{\boldsymbol{S}}_{1,2}. \end{cases} \tag{5.7}$$

In physical variables the Lax matrices are given by

$$L_0^{(1)} = \begin{pmatrix} p_x + (2\lambda - x) p_y & \lambda^2 - \lambda x + \frac{1}{4}(x^2 - 2y) \\ -4 p_y^2 & -p_x - (2\lambda - x) p_y \end{pmatrix}$$
$$L_0^{(2)} = \begin{pmatrix} p_x + \dfrac{1}{2\lambda} y p_y & \lambda - x - \dfrac{1}{4\lambda} y^2 \\ \dfrac{1}{\lambda} p_y^2 & -p_x - \dfrac{1}{2\lambda} y p_y \end{pmatrix}. \tag{5.8}$$

For the dual Stäckel systems the Lax matrices $\tilde{L}_0^{(1,2)}$ have the form

$$\tilde{L}_0^{(1)} = L_0^{(1)} + \begin{pmatrix} 0 & 0 \\ 4\dfrac{p_x p_y}{x} & 0 \end{pmatrix} = L_0^{(1)} + \tilde{H}_0^{(1)} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$
$$\tilde{L}_0^{(2)} = L_0^{(2)} + \begin{pmatrix} 0 & 0 \\ \dfrac{p_x^2 + p_y^2}{x} & 0 \end{pmatrix} = L_0^{(1)} + \tilde{H}_0^{(2)} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{5.9}$$

By using the property $\{h_t(\lambda), v(q)\} = 0$ of the function $v(q)$ (5.7) we can easily prove equation (4.8) for the dual systems by using the same equation for the system with $v_t = 0$

$$\tilde{f}_t = \{\tilde{H}, f + \tilde{H}\} = f_t = 0.$$

Another consequence of this property is that the function $w(\lambda, p, q)$ in (4.5) is equal to zero.

The spectral curves of the matrices L_0 (5.8) coincides with the initial curves $\mathcal{C}_0^{(1,2)}$ (2.4) at $U = 0$

$$z^2 = H_0^{(1)}\lambda + J_0^{(1)} \qquad z^2 = H_0^{(2)} + \frac{J_0^{(2)}}{\lambda}. \tag{5.10}$$

Here $J_0^{(1,2)}$ are the second integrals of motion (2.2). For the dual systems with the Hamilton functions $\tilde{H}_0^{(1,2)}$ the corresponding spectral curves are equal to

$$z^2 = \tilde{H}_0^{(1)}\lambda^2 + \tilde{J}_0^{(1)} \qquad z^2 = \tilde{H}_0^{(2)}\lambda + \frac{\tilde{J}_0^{(2)}}{\lambda}. \tag{5.11}$$

If for the system with the Hamiltonian $H_0^{(1)}$ the Abel map is in one-to-one correspondence on the curve (5.10), then for the same system on the curve

$$z^2 = e_2\lambda^2 + e_1\lambda + e_0$$

the associated Abel map is not uniquely defined in general. So, on this curve we can introduce the second Stäckel system with the dual Hamiltonian $\tilde{H}_0^{(1)}$.

Let us briefly consider systems with polynomial potentials $U \neq 0$. As an example, we introduce different potentials for the curves $\mathcal{C}^{(1,2)}$ (5.1)

$$U^{(1)}(q_j) = \alpha^2 q_j^5 + \beta q_j^3 \qquad U^{(1)}(q_j) = \alpha^2 q_j^3 + \beta q_j. \tag{5.12}$$

To describe these potentials we have to put $N = 6$ and 4 in (4.11) and have to use the following parametric functions:

$$\phi^{(1)}(\lambda) = -\alpha^2\lambda^5 \qquad \text{and} \qquad \phi^{(2)}(\lambda) = -\alpha^2\lambda^3$$

for the curves $\mathcal{C}^{(1)}$ and $\mathcal{C}^{(2)}$, respectively. For both curves the common function $u(\lambda, q_1, q_2)$ is given by

$$u^{(1,2)} = -\alpha^2(\lambda + 2x). \tag{5.13}$$

Here we restrict ourselves to the presentation of the function u only, the complete Lax matrices $L(\lambda)$ may be constructed by the rule (4.6).

The spectral curves of the corresponding matrices (4.6) coincide with the initial curves (3.9). For instance, curves for the systems with dual Hamiltonians $\tilde{H}^{(1,2)}$ are

$$\mathcal{C}^{(1)}: \quad z^2 = \alpha^2\lambda^5 + \beta\lambda^3 - \tilde{H}\lambda - \tilde{J}$$
$$\mathcal{C}^{(2)}: \quad z^2 = \alpha^2\lambda^3 - \tilde{H}\lambda + \beta - \frac{\tilde{J}}{\lambda}.$$

The Poisson bracket relations for the Lax matrix (5.8) and (5.9) are closed into the following linear r-matrix algebra:

$$\{\overset{1}{L}(\lambda),\overset{2}{L}(\mu)\} = [r_{12}(\lambda,\mu),\overset{1}{L}(\lambda)] - [r_{21}(\lambda,\mu),\overset{2}{L}(\mu)]$$
$$r_{21}(\lambda,\mu) = -\Pi r_{12}(\mu,\lambda)\Pi. \tag{5.14}$$

Here the standard notations are introduced:

$$\overset{1}{L}(\lambda) = L(\lambda)\otimes I \qquad \overset{2}{L}(\mu) = I\otimes L(\mu)$$

and Π is the permutation operator of auxiliary spaces [12].

At $v_t = 0$ for the systems related to the matrices $\boldsymbol{S}^{1,2}$ the corresponding r-matrices $r_{ij}(\lambda,\mu)$ in (5.14) may be factorized

$$r_{ij} = r_{ij}^{p} + r_{ij}^{u}. \tag{5.15}$$

The first matrix is a standard r-matrix on the loop algebra $\mathcal{L}(sl(2))$

$$r_{12}^{p}(\lambda,\mu) = \frac{\Pi}{\lambda-\mu} = \frac{1}{\lambda-\mu}\begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}. \tag{5.16}$$

The second matrix may be associated with outer automorphism of the space of infinite-dimensional representations of the underlying algebra $sl(2)$ [5, 11]. The corresponding dynamical r_{ij}^{u}-matrices depend on the coordinates only

$$r_{12}^{u} = \frac{u(\lambda,q)-u(\mu,q)}{\lambda-\mu}\sigma_-\otimes\sigma_- \qquad \sigma_- = \begin{pmatrix} 0 & 0\\ 1 & 0\end{pmatrix}. \tag{5.17}$$

At $v_t \neq 0$ for the dual Stäckel systems related to the matrices $\tilde{\boldsymbol{S}}_{1,2}$ we have to add to the r-matrices (5.15) the third term

$$r_{12}^{v} = v(q_1,q_2)\begin{pmatrix} 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix}. \tag{5.18}$$

This matrix r_{ij}^{v} may be connected with the Drinfeld twist for the Toda lattice associated with the root system $\mathcal{D}_n$. Let us consider the Drinfeld twist [13] of the quantum R-matrix

$$\tilde{R} = FRF_{21}^{-1} \qquad F_{21} = \Pi F\Pi. \tag{5.19}$$

Here the matrix R satisfies the Yang–Baxter equation and the matrix F has the special property [13]. To introduce the corresponding linear r-matrix [14], one obtains

$$R = I + 2\eta r^{p} + \mathrm{O}(\eta^2) \qquad F = I + \eta r^{v} + \mathrm{O}(\eta^2).$$

Then we consider the limit of the twisted matrix $\tilde{R}$ by $\eta\to 0$

$$\tilde{R}_{12} = I + \eta\big(r_{12}^{p} + r_{12}^{v} - \Pi(r_{12}^{p} + r_{12}^{v})\Pi\big) + \mathrm{O}(\eta^2). \tag{5.20}$$

Formally, coefficients by η may be called the twisted linear r-matrix.

By using generators $\boldsymbol{h},\boldsymbol{e},\boldsymbol{f}$ of the underlying Lie algebra $sl(2)$

$$[\boldsymbol{h},\boldsymbol{e}] = 2\boldsymbol{e} \qquad [\boldsymbol{h},\boldsymbol{f}] = -2\boldsymbol{f} \qquad [\boldsymbol{e},\boldsymbol{f}] = \boldsymbol{h} \tag{5.21}$$

let us introduce an appropriate element $\mathcal{F} \in U(sl(2)) \otimes U(sl(2))$

$$\mathcal{F}_\xi = \exp(\xi \cdot \boldsymbol{e} \otimes \boldsymbol{f}) \qquad \xi \in \mathbb{C}$$

from the tensor product of the corresponding universal enveloping algebras $U(sl(2))$ [13]. In the fundamental spin-$\frac{1}{2}$ representation $\rho_{1/2}$ we have

$$F(\xi) = (\rho_{1/2} \otimes \rho_{1/2})\mathcal{F}_\xi = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & \xi & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

To substitute in (5.19) the Yang solution of the Yang–Baxter equation $R = I + (\eta/\lambda)\Pi$ we obtain a twisted R-matrix. If the element $\xi(q)$ is a suitable function on coordinates, this dynamical twisted R-matrix may be used to describe the Toda lattice associated with the $\mathcal{D}_n$ root system [15].

Let us consider the twisted dynamical matrix (5.19) at $\xi = v(q)$. We can see that the linear r-matrix associated with the dual Stäckel system (5.16)–(5.18)

$$r_{12} = r_{12}^p + r_{12}^v \qquad r_{21} = -\Pi(r_{12}^p + r_{12}^v)\Pi$$

is equal to half of the twisted linear matrix (5.20).

Recall, for the Stäckel matrices $\boldsymbol{S}_1$ (5.3) and $\boldsymbol{S}_2$ (5.4) that the corresponding differentials (3.11) span $\mathcal{H}_1$. The associated Hamilton functions have a natural form in physical variables. For instance, Hamiltonians with potentials (5.12) are given by

$$\begin{aligned} H^{(1)} &= 2p_x p_y + \tfrac{1}{4}\alpha^2(y^2 + 5x^2 y + \tfrac{5}{4}x^4) + \tfrac{1}{2}\beta(\tfrac{3}{2}x^2 + y) \\ H^{(2)} &= \tfrac{1}{2}p_x^2 + \tfrac{1}{2}p_y^2 + \tfrac{1}{2}\alpha^2 x(2x^2 + y^2) + \beta. \end{aligned} \tag{5.22}$$

To consider the dual Stäckel systems we have to use the additional transformation

$$x = \sqrt{2\tilde{x}} \qquad p_x = \tilde{p}_x\sqrt{2\tilde{x}} \tag{5.23}$$

for the first curve and the following more complicated transformation:

$$\begin{aligned} x &= \tfrac{3}{2}\tilde{x}^{2/3} \qquad & p_x &= \tilde{p}_x\tilde{x}^{1/3} \\ y &= \sqrt{\frac{2}{3}}\frac{\tilde{p}_y}{\alpha} \qquad & p_y &= -\sqrt{\frac{3}{2}}\alpha\tilde{y} \end{aligned} \tag{5.24}$$

for the second curve. After this canonical change of variables the Hamiltonians $\tilde{H}^{(1,2)}$ (5.5) obtain the natural form

$$\begin{aligned} \tilde{H}^{(1)} &= 2\tilde{p}_x p_y + \frac{\alpha^2(y^2 + 10y\tilde{x} + 5\tilde{x}^2)}{8}\sqrt{\frac{2}{\tilde{x}}} + \frac{\beta(3\tilde{x} + y)}{4}\sqrt{\frac{2}{\tilde{x}}} \\ \tilde{H}^{(2)} &= \tfrac{1}{2}(\tilde{p}_x^2 + \tilde{p}_y^2) + \tfrac{3}{4}\alpha^2\tilde{x}^{-2/3}(\tfrac{9}{2}\tilde{x}^2 + \tilde{y}^2) + \beta\tilde{x}^{-2/3}. \end{aligned} \tag{5.25}$$

The system with the Hamiltonian $H^{(2)}$ is the so-called second integrable case of the Henon–Heiles system [1]. The dual system with the Hamiltonian $\tilde{H}^{(2)}$ is the so-called Holt-type system [1].

An additional canonical transformation (5.24) allows us to obtain natural Hamiltonians for the restricted class of potentials U (5.12) only. Unlike the canonical transformation (5.23) may be used for any potentials U. As an example, the rational potential

$$U(q) = \frac{\alpha}{q^2} + \frac{\beta}{q} + \gamma q + \delta q^2 + \rho q^4$$

gives rise to the following Hamiltonian:

$$\tilde{H} = 2\tilde{p}_x p_y - \frac{4\alpha}{(\tilde{x}-y)^2} - \frac{\beta}{\tilde{x}-y}\sqrt{\frac{2}{\tilde{x}}} + \frac{\gamma}{4}\sqrt{\frac{2}{\tilde{x}}} + \frac{1}{2}\delta + \frac{\rho}{2}(\tilde{x}+y).$$

Also we can add potential terms (5.25) to this Hamiltonian.

At $v = 1$ and $w = 0$ the Lax representation (4.5) for a system with an arbitrary number n of degrees of freedom may be regarded as a generic point at the loop algebra $\mathcal{L}(sl(2))$ in the fundamental representation after an appropriate completion [5]. As an example, for the generalized parabolic coordinate systems function $e(\lambda)$ is given by

$$e(\lambda) = \frac{\prod_{j=1}^{n}(\lambda - q_j)}{\prod_{k=1}^{n-1}(\lambda-\delta_k)} = \lambda - x_n + \sum_{k=1}^{n-1}\frac{x_k^2}{4(\lambda-\delta_k)} \qquad \delta_k \in \mathbb{R}.$$

To construct the Lax representation for a potential motion we can use the outer automorphism of the space of infinite-dimensional representations of $sl(2)$ proposed in [11].

At $v_t \neq 0$ for the dual Stäckel systems the Lax representations may be constructed without any problem as well. For instance, let us consider a system with three degrees of freedom. To construct the Lax matrix (4.5) and (4.6) with the function u given by (5.13) one obtains

$$e(\lambda) = \lambda - x - \frac{y^2}{\lambda} - \frac{z^2}{4(\lambda-k)} \qquad k \in \mathbb{R}$$

$$\tilde{H} = \frac{1}{x}\left(p_x^2 + p_y^2 + p_z^2 + \tfrac{1}{4}a^2kz^2\right) + \tfrac{1}{2}a^2\left(2x^2+y^2+z^2\right).$$

After an additional canonical transformation (5.24) extended on the p_z, z variables the Hamilton function takes the form

$$\tilde{H} = \tilde{p}_x^2 + \tilde{p}_y^2 + \tilde{p}_z^2\left(1 + \tfrac{1}{3}k\tilde{x}^{-2/3}\right) + \tfrac{3}{8}\alpha\tilde{x}^{-2/3}\left(\tfrac{9}{2}\tilde{x}^2 + \tilde{y}^2 + \tilde{z}^2\right).$$

So, the main unsolved problem is to introduce an additional canonical transformation, which transforms the dual Hamilton function $\tilde{H}$ into the natural form.

5.2. Elliptic and polar coordinates ($w(p,q) \neq 0$)

Recall, that the polar coordinate system may be obtained from elliptic coordinate system and, therefore, we shall consider elliptic coordinate systems in detail.

For the elliptic coordinate systems the algebraic curve is given by

$$\mathcal{C}^{(3)} \qquad z^2 = \frac{\prod_{i=1}^{2g+1}(\lambda-\lambda_i)}{(\lambda-k)(\lambda+k)} \qquad k \in \mathbb{C}.$$

Let us consider two Stäckel matrices associated with this curve

$$\boldsymbol{S}_3 = \begin{pmatrix} \dfrac{q_1}{q_1^2-k^2} & \dfrac{q_2}{q_2^2-k^2} \\ \dfrac{1}{q_1^2-k^2} & \dfrac{1}{q_2^2-k^2} \end{pmatrix} \qquad \tilde{\boldsymbol{S}}_3 = \begin{pmatrix} \dfrac{4q_1^2}{q_1^2-k^2} & \dfrac{4q_2^2}{q_2^2-k^2} \\ \dfrac{1}{q_1^2-k^2} & \dfrac{1}{q_2^2-k^2} \end{pmatrix}. \tag{5.26}$$

For the polar coordinate system the Stäckel matrices are non-homogeneous matrices

$$\boldsymbol{S}_4 = \begin{pmatrix} 1 & 0 \\ \dfrac{1}{q_1^2} & \dfrac{1}{4(q_2^2-k)} \end{pmatrix} \qquad \tilde{\boldsymbol{S}}_4 = \begin{pmatrix} q_1^2 & 0 \\ \dfrac{1}{q_1^2} & \dfrac{1}{4(q_2^2-k)} \end{pmatrix}. \tag{5.27}$$

At $U = 0$ the initial hyperelliptic curves (3.9) for the matrices $\boldsymbol{S}_3$ and $\tilde{\boldsymbol{S}}_3$ are given by

$$z^2 = \frac{H_0\lambda + J}{\lambda - k^2} \qquad z^2 = \frac{4\tilde{H}_0\lambda^2 + \tilde{J}_0}{\lambda - k^2} \tag{5.28}$$

with the following Hamiltonians:

$$H_0^{(3)} = \frac{p_1^2(q_1^2 - k^2) - p_2^2(q_2^2 - k^2)}{q_1 - q_2} \qquad \tilde{H}_0^{(3)} = \frac{1}{4(q_1 + q_2)} H. \tag{5.29}$$

The Hamiltonians related to the matrices $\boldsymbol{S}_4$ and $\tilde{\boldsymbol{S}}_4$ read as

$$H_0^{(4)} = p_1^2 - 4\frac{q_2^2 - k}{q_1^2} p_2^2 \qquad \tilde{H}_0^{(4)} = q_1^{-2} H_0^{(4)}. \tag{5.30}$$

Let us fix elliptic coordinates by using the equation

$$e(\lambda) = \frac{(\lambda - q_1)(\lambda - q_2)}{(\lambda - k)(\lambda + k)} = 1 - \frac{x^2}{4(\lambda - k)} - \frac{y^2}{4(\lambda + k)}$$

such that

$$q_1 = \tfrac{1}{8}(x^2 + y^2) + \tfrac{1}{2}\sqrt{(x^2 + y^2)^2 + 16k(x^2 - y^2) + 64k^2}$$
$$q_2 = \tfrac{1}{8}(x^2 + y^2) - \tfrac{1}{2}\sqrt{(x^2 + y^2)^2 + 16k(x^2 - y^2) + 64k^2}.$$

The corresponding equation for the polar coordinates

$$e(\lambda) = \frac{q_1(\lambda - q_2)}{\lambda(\lambda - 1)} = \frac{x^2}{4\lambda} + \frac{4y^2}{\lambda - 1}$$

immediately yields

$$q_1 = r = \sqrt{x^2 + y^2} \qquad q_2 = \cos^2(\phi) = \frac{x^2}{x^2 + y^2}.$$

In physical variables the Hamiltonians (5.29) and (5.30) have a common form

$$H = p_x^2 + p_y^2 \qquad \tilde{H} = \frac{p_x^2 + p_y^2}{x^2 + y^2}.$$

To construct the Lax representations we begin with the calculations of the functions $v(\lambda, q)$ by the rule (4.3)–(4.12)

$$v = \begin{cases} 1 & \text{for matrices } \quad \boldsymbol{S}_{3,4} \\ \frac{1}{4}(q_1 + q_2)^{-1} = \dfrac{1}{x^2 + y^2} & \text{for matrix } \quad \tilde{\boldsymbol{S}}_3 \\ \dfrac{1}{q_1^2} = \dfrac{1}{x^2 + y^2} & \text{for matrix } \quad \tilde{\boldsymbol{S}}_4. \end{cases} \tag{5.31}$$

So, for the Stäckel systems associated with the matrices $\boldsymbol{S}_3$ (5.26) and $\boldsymbol{S}_4$ (5.27) one obtains

$$L_0(\lambda) = \begin{pmatrix} \dfrac{xp_x}{2(\lambda - k)} + \dfrac{yp_y}{2(\lambda + k)} & \epsilon\lambda - \dfrac{x^2}{4(\lambda - k)} - \dfrac{y^2}{4(\lambda + k)} \\ \dfrac{p_x^2}{\lambda - k} + \dfrac{p_y^2}{\lambda + k} & -\dfrac{xp_x}{2(\lambda - k)} - \dfrac{yp_y}{2(\lambda + k)} \end{pmatrix}. \tag{5.32}$$

Here $\epsilon = 1$ for the elliptic coordinate system and $\epsilon = 0$ for the parabolic coordinate system. The spectral curve of the Lax matrix $L_0(\lambda)$ coincides with the initial curve (5.28).

For the dual systems, in contrast to the Cartesian and parabolic coordinates, the Lax matrices have a more complicated form. Both these Lax matrices may be constructed by the rule (4.6) with the following common function $w(p,q)$:

$$w(p,q) = 2\sqrt{\tilde{H}}. \tag{5.33}$$

The Lax matrix reads as

$$\tilde{L}_0(\lambda) = L_0(\lambda) + \begin{pmatrix} we(\lambda) & 0 \\ -2w[h(\lambda) - we(\lambda) - \epsilon w] & -we(\lambda) \end{pmatrix} \qquad \epsilon = 0, 1. \tag{5.34}$$

Here $e(\lambda)$ and $h(\lambda)$ are entries of the corresponding matrices $L_0(\lambda)$ (5.32) at $\epsilon = 0, 1$. As above, the spectral curve of the Lax matrix $\tilde{L}_0(\lambda)$ at $\epsilon = 1$ coincides with the initial curve (5.28).

For the Cartesian and parabolic coordinate systems we can obtain the equation

$$\left\{\left\{H, v^{-1}(q)\right\}e(\lambda, q)\right\} = \{\{H, (q_1 + q_2)\}e(\lambda, q)\} = 2$$

on the Hamiltonian H, the functions $e(\lambda)$ and $v(q)$ define the change of time. For the polar and elliptic coordinate systems the corresponding equation is

$$\left\{\left\{H, v^{-1}(q)\right\}e(\lambda, q)\right\} = 8(e(\lambda) - \epsilon) \qquad \epsilon = 0, 1.$$

Hence, from equation (4.8) it follows that the function $w(p,q)$ in (4.5)–(5.33) does not equal zero. If we consider a more complicated change of the time for the Cartesian and parabolic coordinate systems, one obtains a non-zero function w (4.5) as well.

The Poisson bracket relations for the Lax matrix $L_0(\lambda)$ (5.32) are closed into the standard linear r-matrix algebra (5.14) with the rational r-matrix (5.16) on the loop algebra $\mathcal{L}(sl(2))$ [12].

The Poisson bracket relations for the dual Lax matrix $\tilde{L}_0(\lambda)$ (5.34) have a polylinear form

$$\left\{\overset{1}{L}(\lambda)\overset{2}{L}(\mu)\right\} = \left[r_{12}\overset{1}{L}(\lambda)\right] + \left[r_{21}\overset{2}{L}(\mu)\right]$$
$$+R\overset{1}{L}(\lambda)\overset{2}{L}(\mu) + \overset{1}{L}(\lambda)\overset{2}{L}(\mu)R - \overset{1}{L}(\lambda)R\overset{2}{L}(\mu) - \overset{2}{L}(\mu)R\overset{1}{L}(\lambda).$$

Here the linear r-matrix reads as

$$r_{12}(\lambda, \mu) = r^p_{12}(\lambda - \mu) + 4\epsilon r^w_{12} \qquad r_{21}(\lambda, \mu) = -\Pi r_{12}(\lambda, \mu)\Pi$$

where $r^p(\lambda - \mu)$ is the standard linear r-matrix on the loop algebra $\mathcal{L}(sl(2))$. The second dynamical term is given by

$$r^w_{12} = v(q)\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ w & 1 & 0 & 0 \\ 0 & -w & 0 & 0 \end{pmatrix}.$$

The quadratic R-matrix is closed to the twisted linear r-matrix (5.20)

$$R = -\frac{2}{w}\left(r^w_{12} + r^w_{21}\right) = -\frac{2}{w}\left(r^w_{12} - \Pi r^w_{12}\Pi\right).$$

At $U \neq 0$ functions $u(\lambda, q)$ may be constructed as usual [5, 11]. Note, that both dual Hamiltonians obtain a natural form after the following additional canonical transformation of variables:

$$x = \sqrt{\tilde{x}} - \sqrt{\tilde{y}} \qquad p_x = \sqrt{\tilde{x}}\,\tilde{p}_x - \sqrt{\tilde{y}}\,\tilde{p}_y$$
$$y = -\mathrm{i}(\sqrt{\tilde{x}} + \sqrt{\tilde{y}}) \qquad p_y = \mathrm{i}(\sqrt{\tilde{x}}\,\tilde{p}_x + \sqrt{\tilde{y}}\,\tilde{p}_y).$$

As an example, for an elliptic coordinate system the uniform potential

$$U^{(3)}(q_j) = \alpha q_j^2 + \beta q_j$$

gives rise to the following dual Hamiltonian:

$$\tilde{H} = 2\tilde{p}_x\tilde{p}_y + \frac{\alpha}{4}(\tilde{x}+k)(\tilde{y}+k) - \frac{\beta}{8\sqrt{\tilde{x}\tilde{y}}}(2\tilde{x}\tilde{y} + k\tilde{x} + k\tilde{y}).$$

For the polar coordinate system we present the non-uniform degenerate potentials

$$U_1^{(4)}(q_1) = \beta \qquad U_2^{(4)}(q_2) = 0$$

associated with the following dual Hamiltonian:

$$\tilde{H} = \frac{\tilde{p}_x\tilde{p}_y}{2} - \beta\left(\frac{16}{\sqrt{\tilde{x}\tilde{y}}} + \frac{1}{\tilde{x}} + \frac{1}{\tilde{y}} - \frac{2}{\tilde{x}\tilde{y}}\right).$$

Both of these systems may be considered as integrable deformations of the Kepler problem.

6. Conclusions

In this paper we have considered relations between the different Stäckel systems. The proposed change of the time is related to the ambiguity of the Abel map for the hyperelliptic curves. Of course, a particular family of such transformations (2.8) does not exhaust the set of canonical changes of time, which preserve integrability. As an example, the Kolosoff transformation $\{t, p, q\} \to \{\tilde{t}, \tilde{p}, q\}$ [2] connects the Stäckel system with the other integrable non-Stäckel system. So, it would be interesting to investigate integrable systems connected with the Stäckel systems by canonical transformations of time.

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